ON THE CHARACTERISTIC POLYNOMIAL OF THE ALMOST MATHIEU OPERATOR

MICHAEL P. LAMOUREUX AND JAMES A. MINGO

ABSTRACT. Let A_θ be the rotation C*-algebra for angle θ. For $\theta = p/q$ with p and q relatively prime, A_θ is the sub-C*-algebra of $M_q(C(\mathbb{T}^2))$ generated by a pair of unitaries u and v satisfying $uv = e^{2\pi i\theta}vu$. Let $h_{\theta,\lambda} = u+u^{-1}+\lambda/2(v+v^{-1})$ be the almost Mathieu operator. By proving an identity of rational functions we show that for q even, the constant term in the characteristic polynomial of $h_{\theta,\lambda}$ is $(-1)^{q/2}(1+(\lambda/2)^q) - (z_1^q + z_1^{-q} + (\lambda/2)^q(z_2^q + z_2^{-q}))$.

1. Introduction

Let θ, λ, and ψ be real numbers with λ positive. The second order difference operator $H_{\theta,\lambda,\psi}$ on $\ell^2(\mathbb{Z})$ given by

$$H_{\theta,\lambda,\psi}(\xi)(n) = \xi(n+1) + \xi(n-1) + \lambda\cos(2\pi n\theta + \psi)\xi(n)$$

for $\xi \in \ell^2(\mathbb{Z})$ is called the almost Mathieu operator. $H_{\theta,\lambda,\psi}$ is a discrete Schrödinger operator which models an electron moving in a crystal lattice in a plane perpendicular to a magnetic field.

An object of much study has been the spectrum $\sigma(\theta,\lambda) = \cup_\psi \sigma(H_{\theta,\lambda,\psi})$. In [H], Hofstadter calculated $\sigma(\theta, 2)$ for $\theta = p/q$ and $1 \le p < q \le 50$. The remarkable pattern he found is called Hofstadter's butterfly. For irrational θ, a long standing concern has been the connectedness and Lebesgue measure of $\sigma(\theta,\lambda)$ and the labelling of the gaps, about which quite a bit is now known (see [AJ], [AK], and [P] for spectacular recent advances as well as [AvMS], [BS], [B], [CEY], [LT] for earlier work). In addition there has been numerical work on computing the spectrum to high accuracy for large q [A1, A2, L].

Let A_θ be the rotation C*-algebra (see [B]). For $\theta = p/q$ with p and q relatively prime and $\rho = e^{2\pi\theta}$ let

$$u_\theta = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix} \text{ and } v_\theta = \begin{pmatrix} \rho & & & & \\ & \rho^2 & & & \\ & & \ddots & & \\ & & & \rho^{q-1} & \\ & & & & 1 \end{pmatrix}$$

i.e u_θ cyclically permutes the elements of the standard basis and v_θ is a diagonal operator. Then define $u, v : \mathbb{T}^2 \to M_q(\mathbb{C})$ by $u(z_1, z_2) = z_1 u_\theta$ and $v(z_1, z_2) = z_2 v_\theta$. Then $u\,v = \rho v\,u$ and A_θ is the C*-algebra generated by u and v (see [B]). The

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operator $h_{\theta,\lambda} = u + u^{-1} + \lambda/2(v + v^{-1})$ contains all the spectral information of $H_{\theta,\lambda,\psi}$ in that $\mathrm{Sp}(h_{\theta,\lambda}) = \sigma_{\theta,\lambda} := \cup_\psi \mathrm{Sp}(H_{\theta,\lambda,\psi})$.

The main tool in the analysis of $\sigma_{\theta,\lambda}$ is $\Delta_{\theta,\lambda}$, the discrete analogue of the discriminant. For $\theta = p/q$, $\Delta_{\theta,\lambda}(x) = \mathrm{Tr}(A_1(x) \cdots A_q(x))$ where

$$A_k(x) = \begin{pmatrix} x - \lambda\cos(2\pi kp/q + \pi/(2q)) & -1 \\ 1 & 0 \end{pmatrix}$$

Below are the first few values of this polynomial. Note the form of $\Delta_{\theta,\lambda}$ so displayed depends only on the denominator q; however, $\xi_\theta = 2\cos(2\pi p/q)$ depends on the numerator p.

q	$\Delta_{\theta,2}(x)$ for $\theta = p/q$ and $\xi_\theta = 2\cos(2\pi\theta)$
2	$x^2 - 4$
3	$x^3 - 6x$
4	$x^4 - 8x^2 + 4$
5	$x^5 - 10x^3 + 5(3 - \xi_\theta)x$
6	$x^6 - 12x^4 + 6(5 - \xi_\theta)x^2 - 4$
7	$x^7 - 14x^5 + 7(7 - \xi_\theta)x^3 - 7(6 - 2\xi_\theta + 2\xi_{2\theta})x$
8	$x^8 - 16x^6 + 8(9 - \xi_\theta)x^4 - 8(12 - 4\xi_\theta + 2\xi_{2\theta})x^2 + 4$
9	$x^9 - 18x^7 + 9(11 - \xi_\theta)x^5 - 9(31/3 - 6\xi_\theta + 2\xi_{2\theta})x^3 + 9(14 - 8\xi_\theta + 3\xi_{2\theta})x$

One can calculate for $k = 1, 2$ the coefficient of x^{q-2k}, for $k = 3$ the formula is conjectural (from numerical evidence). A deeper understanding of the structure of $\Delta_{\theta.\lambda}$ would be quite interesting.

k	coefficient of x^{q-2k} in $\Delta_{\theta,\lambda}$ $\quad(\mu = \lambda/2)$
1	$-q(1 + \mu^q)$
2	$q(\frac{1}{q-2}\binom{q-2}{2}\mu^4 + (q - 4 - \xi_\theta)\mu^2 + \frac{1}{q-2}\binom{q-2}{2})$
3	$-q(\frac{1}{q-3}\binom{q-3}{3}\mu^6 + (1 + \binom{q-5}{2} - (q-6)\xi_\theta + \xi_{2\theta})\mu^4$
	$+ (1 + \binom{q-5}{2} - (q-6)\xi_\theta + \xi_{2\theta})\mu^2 + \frac{1}{q-3}\binom{q-3}{3})$

The connection with the characteristic polynomial of $h_{\theta,\lambda}$ is given by

$$\det(xI_q - h_{\theta,\lambda}(z_1, z_2)) = \Delta_{\theta,\lambda}(x) + z_1^q + z_1^{-q} + (\lambda/2)^q(z_2^q + z_2^{-q}) \tag{1}$$

and thus $\sigma_{\theta,\lambda} = \Delta_{\theta,\lambda}^{-1}[-2(1+(\lambda/2)^q),\ 2(1+(\lambda/2)^q)]$. Indeed, $\Delta_{\theta,\lambda}(x)$ can be written as a determinant (c.f. Toda [T, §4])

$$\Delta_{p/q,\lambda}(x) = \det\begin{pmatrix} \alpha_1 & 1 & & & 1 \\ 1 & \alpha_2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & \\ & & & & 1 \\ 1 & & & 1 & \alpha_q \end{pmatrix} + 2\Big\{(-1)^q + (\lambda/2)^q\Big\} \tag{2}$$

where all the other entries are 0 and $\alpha_k = x - \lambda\cos(2\pi kp/q + \pi/(2q))$. Since

$$\Delta_{p/q,\lambda}(-x) = (-1)^q\Delta_{p/q,\lambda}(x) \tag{3}$$

the coefficient of $x^{q-(2k+1)}$ is 0 for $0 \le k < q/2$.

The main result of the paper asserts that for $a_l = 2\cos(2\pi lp/q)$ and $1 \le k < q/2$ we have

$$\sum_{i_1,i_2,\dots,i_{q-2k}} a_{i_1}a_{i_2}\cdots a_{i_{q-2k}} = 0$$

where the summation is over all subsets of $\{1,2,3,\dots,q\}$ obtained by deleting k pairs of adjacent elements – counting 1 and k as adjacent. This is proved by establishing the following identity for $k \ge 3$ and $q \ge 2k-1$

$$\sum_{i_1=1}^{q-2(k-1)} \cdots \sum_{i_k=i_{k-1}+2}^{q} \prod_{j=1}^{k} \frac{(x^{-i_j}+x^{i_j})^{-1}}{(x^{-i_j-1}+x^{i_j+1})} = \frac{(x^{-q}-x^{q})\prod_{i=k+1}^{2k-2}(x^{-q+i}-x^{q-i})}{\prod_{i=1}^{k}(x^{-2i}-x^{2i})\prod_{i=-1}^{k-2}(x^{-q+i}+x^{q-i})}$$

$$+\frac{(x^{-1}+x^{1})^{-1}(x^{-q}+x^{q})^{-1}}{(x^{-2}+x^{2})(x^{-q-1}+x^{q+1})}\sum_{i_1=3}^{q-2(k-2)}\cdots\sum_{i_{k-2}=i_{k-3}+2}^{q-2}\prod_{j=1}^{k-2}\frac{(x^{-i_j}+x^{i_j})^{-1}}{(x^{-i_j-1}+x^{i_j+1})}$$

We then use this to show that for $a_l = 2\cos(2\pi lp/q)$

$$\det\begin{pmatrix} a_1 & 1 & & & 1\\ 1 & a_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1\\ 1 & & & 1 & a_q \end{pmatrix} = \begin{cases} 0 & q\equiv 0 \pmod 4\\ 4 & q \equiv 1,3 \pmod 4\\ -8 & q\equiv 2 \pmod 4\end{cases}$$

From this we show that the constant term (i.e. the coefficient of x^0) in $\det(xI_q - h_{\theta,\lambda}(z_1,z_2))$ is

$$(-1)^{q/2}2(1+(\lambda/2)^q)) - (z_1^q + z_1^{-q} + (\lambda/2)^q(z_2^q+z_2^{-q}))$$

when q is even. When q is odd it follows from (3) that the coefficient of x^0 is $-(z_1^q+z_1^{-q}+(\lambda/2)^q(z_2^q+z_2^{-q}))$.

Similar, though simpler, reasoning shows that the coefficient of x^{q-2} is $-q(1+\lambda/2)$ and that the coefficient of x^{q-4} is $(\lambda/2)^4q(q-3)/2+(\lambda/2)^2q(q-4-2\cos(2\pi\theta))+q(q-3)/2$.

2. The Main Theorem

Let us use the following notation: let $a_1,\dots,a_n$ be elements of a commutative ring and let

$$(a_1,a_2,\dots,a_n) = \begin{vmatrix} a_1 & 1 & & & 0\\ 1 & a_2 & 1 & & \\ & \ddots & a_3 & \ddots & \\ & & \ddots & \ddots & 1\\ 0 & & & 1 & a_n \end{vmatrix}$$

and

$$[\![a_1, a_2, \ldots, a_n]\!] = \begin{vmatrix} a_1 & 1 & & & 1 \\ 1 & a_2 & 1 & & \\ & \ddots & a_3 & \ddots & \\ & & \ddots & \ddots & 1 \\ 1 & & & 1 & a_n \end{vmatrix}.$$

The first matrix is a tridiagonal matrix with 1's on the sub and super-diagonal and 0's elsewhere. The second matrix is the same tridiagonal matrix with in addition 1's in the upper right and lower left corners, all other entries are 0. Expanding along the bottom row we have

$$[\![a_1, a_2, \ldots, a_n]\!] = (a_1, a_2, \ldots, a_n) - (a_2, a_3, \ldots, a_{n-1}) + 2(-1)^{n-1} \tag{4}$$

and

$$[\![-a_1, -a_2, \ldots, -a_n]\!] = (-1)^n [\![a_1, a_2, \ldots, a_n]\!] + 2(-1)^{n-1}. \tag{5}$$

Rewriting equation (2) we have

$$\Delta_{p/q,\lambda}(x) = [\![a_1, \ldots, a_q]\!] + 2((-1)^q + (\lambda/2)^q) \tag{6}$$

Notation 2.1. *(i)* For $0 \le k \le n/2$, let $S\left[{n \atop k}\right] = \{I \subset \{1, 2, \ldots, n\} \mid |I| = n - 2k$ and I is obtained from $\{1, 2, \ldots, n\}$ by deleting k pairs of adjacent elements$\}$. $S\left[{2k \atop k}\right] = \{\emptyset\}$, $S\left[{2k+1 \atop k}\right] = \big\{\{1\}, \{3\}, \{5\}, \ldots, \{2k+1\}\big\}$, $\ldots$, $S\left[{n \atop 0}\right] = \Big\{\{1, 2, 3, \ldots, n\}\Big\}$.

(ii) For $0 \le k \le (n-1)/2$, let $S'\left[{n \atop k}\right] = \{I \subset \{2, 3, \ldots, n\} \mid |I| = n - 2k - 1$ and I is obtained from $\{2, 3, \ldots, n\}$ by deleting k pairs of adjacent elements$\}$. $S'\left[{2k+1 \atop k}\right] = \emptyset$, $S'\left[{2k+2 \atop k}\right] = \big\{\{2\}, \{4\}, \{6\}, \ldots, \{2k+2\}\big\}$, $\ldots$, $S'\left[{n \atop 0}\right] = \{2, 3, \ldots, n\}$.

(iii) For $\mathcal{S}$ a collection of subsets of $\{1, 2, \ldots, n-1\}$ let $\mathcal{S} \vee \{n\} = \{I \cup \{n\} \mid I \in \mathcal{S}\}$.

(iv) For $0 \le k \le n/2$, let $\widetilde{S}\left[{n \atop k}\right] = \{I \subset \{1, 2, 3, \ldots, n\} \mid |I| = n - 2k$ and I is obtained from $\{1, 2, \ldots, n\}$ by deleting k pairs of adjacent elements, counting $\{n, 1\}$ as an adjacent pair$\}$. $\widetilde{S}\left[{2k \atop k}\right] = \emptyset$, $\widetilde{S}\left[{2k+1 \atop k}\right] = \big\{\{1\}, \{2\}, \ldots, \{n\}\big\}$, $\ldots$, $\widetilde{S}\left[{n \atop 0}\right] = \{1, 2, 3, \ldots, n\}$.

(v) For $a_1, a_2, a_3, \ldots, a_n$ elements of a commutative ring, and $I = \{i_1, i_2, i_3, \ldots, i_k\} \subset \{1, 2, 3, \ldots, n\}$, let $a_I = a_{i_1} a_{i_2} a_{i_3} \cdots a_{i_n}$. We shall adopt the convention that $a_\emptyset = 1$

Part *(ii)* of the next proposition goes back to Sylvester's original paper on continuants [S]; part *(iv)* is a straightforward extension of this. For the reader's convenience we present a proof.

Proposition 2.2. *(i) Suppose* $1 \le k < n/2$*, then* $S\left[{n \atop k}\right] = \left(S\left[{n-1 \atop k}\right] \vee \{n\}\right) \cup S\left[{n-2 \atop k-1}\right]$.

(ii)

$$\sum_{k=0}^{[n/2]} (-1)^k \sum_{I \in S\left[{n \atop k}\right]} a_I = a_n \sum_{k=0}^{[(n-1)/2]} (-1)^k \sum_{I \in S\left[{n-1 \atop k}\right]} a_I - \sum_{k=0}^{[(n-2)/2]} (-1)^k \sum_{I \in S\left[{n-2 \atop k}\right]} a_I.$$

(iii) $\widetilde{S}\left[{n \atop k}\right] = S\left[{n \atop k}\right] \cup S'\left[{n-1 \atop k-1}\right]$ *for* $1 \leq k \leq n/2$.

(iv) When n *is odd,*

$$\sum_{k=0}^{[n/2]}(-1)^k \sum_{I\in\widetilde{S}\left[{n \atop k}\right]} a_I = \sum_{k=0}^{[n/2]}(-1)^k \sum_{I\in S\left[{n \atop k}\right]} a_I - \sum_{k=0}^{[(n-1)/2]} (-1)^k \sum_{I\in S'\left[{n-1 \atop k}\right]} a_I$$

When n *is even,*

$$\sum_{k=0}^{[n/2]}(-1)^k \sum_{I\in\widetilde{S}\left[{n \atop k}\right]} a_I = \sum_{k=0}^{[n/2]}(-1)^k \sum_{I\in S\left[{n \atop k}\right]} a_I + (-1)^{n/2} - \sum_{k=0}^{[(n-1)/2]} (-1)^k \sum_{I\in S'\left[{n-1 \atop k}\right]} a_I$$

Proof. *(i)* Let $I \in S\left[{n \atop k}\right]$. If $n \notin I$ then $n-1 \notin I$ and so $I \in S\left[{n-2 \atop k-1}\right]$. Suppose $n \in I$. Let $K = \{1, 2, 3, \dots, n\} \setminus I$ and $\dot{I} = I \setminus \{n\}$. Then $\dot{I} = \{1, 2, 3, \dots n-1\} \setminus K$; so $\dot{I} \in S\left[{n-1 \atop k}\right]$. Hence $I = \dot{I} \cup \{n\} \in S\left[{n-1 \atop k}\right] \vee \{n\}$.

(ii) Let us assume that $n = 2m$ is even. The same idea works for odd n but the proof is slightly simpler. Observe

$$\begin{aligned}
\sum_{k=0}^{[n/2]}(-1)^k \sum_{I\in S\left[{n \atop k}\right]} a_I &= \sum_{I\in S\left[{n \atop 0}\right]} a_I + \sum_{k=1}^{m-1}(-1)^k \sum_{I\in S\left[{n \atop k}\right]} a_I + (-1)^m \\
&= \sum_{I\in S\left[{n \atop 0}\right]} a_I + a_n \sum_{k=1}^{m-1}(-1)^k \sum_{I\in S\left[{n-1 \atop k}\right]} a_I + \sum_{k=1}^{m-1}(-1)^k \sum_{I\in S\left[{n-2 \atop k-1}\right]} a_I + (-1)^m \\
&= a_n\left\{ \sum_{I\in S\left[{n-1 \atop 0}\right]} a_I + \sum_{k=1}^{m-1}(-1)^k \sum_{I\in S\left[{n-1 \atop k}\right]} a_I \right\} + \sum_{k=1}^{m-1}(-1)^k \sum_{I\in S\left[{n-2 \atop k-1}\right]} a_I + (-1)^m \\
&= a_n \sum_{k=0}^{m-1}(-1)^k \sum_{I\in S\left[{n-1 \atop k}\right]} a_I - \sum_{k=0}^{m-1}(-1)^k \sum_{I\in S\left[{n-2 \atop k}\right]} a_I \\
&= a_n \sum_{k=0}^{[(n-1)/2]} (-1)^k \sum_{I\in S\left[{n-1 \atop k}\right]} a_I - \sum_{k=0}^{[(n-2)/2]} (-1)^k \sum_{I\in S\left[{n-2 \atop k}\right]} a_I.
\end{aligned}$$

(iii) For $I \in \widetilde{S}\left[{n \atop k}\right]$ let $K_1 = \{1, 2, 3, \dots, n\} \setminus I$ and $K_2 = \{2, 3, \dots, n\} \setminus I$. $\min\{i \mid i \in K_1\}$ is odd if and only if $I \in S\left[{n \atop k}\right]$ and $\min\{i \mid i \in K_2\}$ is even if and only if $I \in S'\left[{n-1 \atop k-1}\right]$

(iv) Suppose $n = 2m$. Then

$$\begin{aligned}
\sum_{k=0}^{[n/2]}(-1)^k \sum_{I\in\widetilde{S}\left[{n \atop k}\right]} a_I &= \sum_{I\in\widetilde{S}\left[{n \atop 0}\right]} a_I + \sum_{k=1}^{m-1}(-1)^k \sum_{I\in\widetilde{S}\left[{n \atop k}\right]} a_I + (-1)^m \sum_{I\in\widetilde{S}\left[{n \atop m}\right]} a_I \\
&= \left(\sum_{I\in S\left[{n \atop 0}\right]} a_I + \sum_{k=1}^{m}(-1)^k \sum_{I\in S\left[{n \atop k}\right]} a_I \right) + \sum_{k=1}^{m}(-1)^k \sum_{I\in S'\left[{n-1 \atop k-1}\right]} a_I + (-1)^m \\
&= \sum_{k=0}^{m}(-1)^k \sum_{I\in S\left[{n \atop k}\right]} a_I - \sum_{k=0}^{m-1}(-1)^k \sum_{I\in S'\left[{n-1 \atop k}\right]} a_I + (-1)^m.
\end{aligned}$$

The case of n odd is similar. $\square$

Corollary 2.3. *Let $a_1, a_2, \ldots, a_n$ be elements of a commutative ring.*

(i) $(a_1, \ldots, a_n) = \sum_{k=0}^{[n/2]} (-1)^k \sum_{I \in S\left[{n \atop k}\right]} a_I$

(ii)

$$[\![a_1, \ldots, a_n]\!] = \begin{cases} \sum_{k=0}^{[n/2]} (-1)^k \sum_{I \in \widetilde{S}\left[{n \atop k}\right]} a_I + 2 & n \text{ odd} \\ \sum_{k=0}^{[n/2]} (-1)^k \sum_{I \in \widetilde{S}\left[{n \atop k}\right]} a_I - 2 + (-1)^{n/2} & n \text{ even} \end{cases}$$

Proof. *(i)* For $n = 1$ the left hand side and the right hand side equal a_1. Both sides satisfy the same recurrence relation.

(ii) By equation (6)

$$[\![a_1, \ldots, a_n]\!] = (a_1, \ldots, a_n) - (a_2, \ldots, a_{n-1}) - (-1)^n 2$$

so the result now follows from (i) and Proposition 2.2 (iii). $\square$

Proposition 2.4. *Let $1 \leq p < q$ be relatively prime, $\theta = p/q$, and $a_k = 2\cos(2\pi k\theta)$. Then*

$$a_1 a_2 \cdots a_q = \begin{cases} 0 & q \equiv 0 \pmod 4 \\ 2 & q \equiv 1, 3 \pmod 4 \\ -4 & q \equiv 2 \pmod 4. \end{cases}$$

Proof. Let T_q be the qth Chebyshev polynomial of the first kind. The constant term of T_q is 0 for q odd and $(-1)^{q/2}$ for q even. The result now follows from the identity (see e.g. [R, §1.2])

$$\prod_{i=1}^{q} (x - a_i) = 2(T_q(x/2) - 1).$$

$\square$

The statement of the main theorem follows. Its proof will be given at the end of the next section.

Theorem 2.5. *Let $1 \leq p < q$ be relatively prime, $a_k = 2\cos(2\pi k\theta)$, and $\theta = p/q$. For $1 \leq k < q/2$,*

$$\sum_{I \in \widetilde{S}\left[{q \atop k}\right]} a_I = 0.$$

Corollary 2.6. *Let $1 \leq p < q$ be relatively prime, $\theta = p/q$, $\lambda > 0$, and $a_k = \lambda\cos(2\pi k\theta)$. Then*

$$[\![a_1, a_2, \ldots, a_q]\!] = \begin{cases} 0 & q \equiv 0 \pmod 4 \\ 2(1 + (\lambda/2)^q) & q \equiv 1, 3 \pmod 4 \\ -4(1 + (\lambda/2)^q) & q \equiv 2 \pmod 4 \end{cases}$$

and $\Delta_{\theta,\lambda}(0) = (-1)^{q/2} 2(1 + (\lambda/2)^q)$ for q even.

Proof. Suppose q is even. By Theorem 2.5 all the terms of

$$\sum_{k=0}^{[q/2]}(-1)^k \sum_{I\in\widetilde{S}\left[{q \atop k}\right]} a_I$$

are zero except the terms for $k=0$ and $k=q/2$. The term for $k=0$ is $a_1a_2\cdots a_q$. The term for $k=q/2$ is $(-1)^{q/2}$. Thus when $q=4m$ we have by Proposition 2.4

$$[\![a_1,a_2,\dots,a_q]\!] = a_1a_2\cdots a_q - (-1)^q2 + (-1)^{q/2}2 = 0,$$

and when $q=4m+2$,

$$[\![a_1,a_2,\dots,a_q]\!] = a_1a_2\cdots a_q - (-1)^q2 + (-1)^{q/2}2 = -4(1+(\lambda/2)^q).$$

To obtain the final claim we apply equation (6). $\square$

From the corollary and equation (1) we have the theorem which corrects an error in [CEY, p. 232].

Theorem 2.7. *The coefficient of* x^0 *in* $\det(xI_q - h_{\theta,\lambda}(z_1,z_2))$ *is*

$$-\left(z_1^q + z_1^{-q} + (\lambda/2)^q(z_2^q+z_2^{-q})\right) + (-1)^{q/2}2(1+(\lambda/2)^q))$$

when q *is even and* $-(z_1^q + z_1^{-q} + (\lambda/2)^q(z_2^q+z_2^{-q}))$ *when* q *is odd.*

3. Proof of the Main Theorem

Theorem 3.1. *Suppose* $a_1,a_2,\dots,a_q$ *are elements in a commutative ring and let* $a_{q+1}=a_1$. *For* $I\subset\{1,2,\dots,q\}$, *let* $I^c=\{1,2,\dots,q\}\setminus I$, *be the complement of* I *in* $\{1,2,\dots,q\}$. *Then*

$$\sum_{I\in\widetilde{S}\left[{q \atop k}\right]} a_{I^c} = \sum_{i_1=1}^{q-2(k-1)}\ \sum_{i_2=i_1+2}^{q-2(k-2)}\cdots\sum_{i_k=i_{k-1}+2}^{q}\ \prod_{j=1}^{k} a_{i_j}a_{i_j+1}$$

$$- a_1a_2\left[\sum_{i_1=3}^{q-2(k-2)}\ \sum_{i_2=i_1+2}^{q-2(k-3)}\cdots\sum_{i_{k-2}=i_{k-3}+2}^{q-2}\ \prod_{j=1}^{k-2} a_{i_j}a_{i_j+1}\right]a_qa_{q+1}.$$

Proof. Recall that elements of $\widetilde{S}\left[{q \atop k}\right]$ are obtained by deleting k adjacent pairs $\{i,i+1\}$ from $\{1,2,\dots,q\}$, counting q and 1 as adjacent. So if $I^c\in\widetilde{S}\left[{q \atop k}\right]$ then $I=\{i_1,j_1,i_2,j_2,\dots,i_k,j_k\}$ with $1\le i_1, j_1=i_1+1<i_2,\dots,j_{k-1}=i_{k-1}+1<i_k\le q$ and either $j_k=i_k+1$ if $i_k<q$ or $j_k=1$ if $i_k=q$.

Now let $T\left[{q \atop k}\right]=\{\{i_1,j_1,i_2,j_2,\dots,i_k,j_k\}\mid 1\le i_1, j_1=i_1+1<i_2,\dots,j_{k-1}=i_{k-1}+1<i_k\le q, j_k=i_k+1\}$. Define $\phi:\{1,2,\dots,q,q+1\}\to\{1,2,\dots,q\}$ by $\phi(q+1)=1$ and $\phi(i)=i$ for $i\le q$. Then $a_{\phi(I)}=a_I$ for $I\in T\left[{q \atop k}\right]$.

If $I=\{i_1,j_1,i_2,j_2,\dots,i_k,j_k\}$ and $i_1=1$ and $i_k=q$ then $\phi(I)^c\notin\widetilde{S}\left[{q \atop k}\right]$ because $\phi(j_k)=\phi(i_1)=1$ and the pairs must be disjoint. So let $T'\left[{q \atop k}\right]=\{\{1,2,i_1,j_1,\dots,i_{k-1},j_{k-1},q,q+1\}\mid 3\le i_1, j_1=i_1+1<i_2,\dots,i_{k-1}\le q-2, j_{k-1}=i_{k-1}+1\}$.

For $I\in T\left[{q \atop k}\right]\setminus T'\left[{q \atop k}\right]$, $\phi(I)^c\in\widetilde{S}\left[{q \atop k}\right]$ and $\phi: T\left[{q \atop k}\right]\setminus T'\left[{q \atop k}\right]\to\widetilde{S}\left[{q \atop k}\right]$ is a bijection. This with the identity $a_{\phi(I)}=a_I$ proves the theorem. $\square$

Lemma 3.2. *(i) For* $q\ge1$,

$$\sum_{i=1}^{q}(x^{-i}+x^i)^{-1}(x^{-i-1}+x^{i+1})^{-1} = \frac{x^{-q}-x^q}{(x^{-2}-x^2)(x^{-q-1}+x^{q+1})}.$$

(ii) For $k \geq 1$

$$\prod_{i=1}^{2k}(x^{-i}+x^{i})^{-1} = \prod_{i=1}^{k} \frac{(x^{-i}-x^{i})}{(x^{-2i}-x^{2i})(x^{-(k+i)}+x^{k+i})}.$$

Proof. *(i)* One checks directly that the formula holds when $q = 1$, then *(i)* follows by induction on q.

(ii) follows from the identity

$$\frac{x^{-i}-x^{i}}{(x^{-2i}-x^{2i})(x^{-k-i}+x^{k+i})} = \frac{1}{(x^{-i}+x^{i})(x^{-k-i}+x^{k+i})}$$

□

Corollary 3.3. *For* $q \geq 5$

$$\frac{(x^{-1}+x)^{-1}(x^{-2}+x^{2})^{-1}}{(x^{-q}+x^{q})(x^{-q-1}+x^{q+1})} \sum_{i=3}^{q-2}(x^{-i}+x^{i})^{-1}(x^{-i-1}+x^{i+1})^{-1}$$

$$= \frac{(x^{-3}-x^{3})(x^{-q+4}-x^{q-4})}{(x^{-4}-x^{4})(x^{-6}-x^{6})(x^{-q+1}+x^{q-1})(x^{-q}+x^{q})(x^{-q-1}+x^{q+1})}$$

Proof. By Lemma 3.2 *(i)*

$$\begin{aligned} &\sum_{i=3}^{q-2}(x+-i+x^{i})^{-1}(x^{-i-1}+x^{i+1})^{-1} \\ &= \frac{x^{-q+2}-x^{q-2}}{(x^{-2}-x^{2})(x^{-q+1}+x^{q-1})} - \frac{x^{-2}-x^{2}}{(x^{-2}-x^{2})(x^{-3}+x^{3})} \\ &= \frac{(x^{-q+4}-x^{q-4})(x^{-1}+x)}{(x^{-2}-x^{2})(x^{-3}+x^{3})(x^{-q+1}+x^{q-1})} \end{aligned}$$

The result then follows by multiplying both sides by $(x^{-1}+x)(x^{-2}+x^{2})(x^{-3}+x^{3})(x^{-q+1}+x^{q-1})$. □

Theorem 3.4. *For* $k \geq 1$ *and* $q \geq 2k-1$,

$$\sum_{i_1=1}^{q-2(k-1)} \sum_{i_2=i_1+2}^{q-2(k-2)} \cdots \sum_{i_k=i_{k-1}+2}^{q} \prod_{j=1}^{k}(x^{-i_j}+x^{i_j})^{-1}(x^{-i_j-1}+x^{i_j+1})^{-1}$$

$$= \frac{\prod_{i=k-1}^{2k-2}(x^{-(q-i)}-x^{q-i})}{\prod_{i=1}^{k}(x^{-2i}-x^{2i})\prod_{i=-1}^{k-2}(x^{-(q-i)}+x^{q-i})} \tag{7}$$

Proof. We prove the equation by induction on k. When $k = 1$ the equation holds by Lemma 3.2 *(i)*. Lemma 3.2 *(ii)* shows that for arbitrary k the formula holds for $q = 2k-1$; so we fix k and proceed by induction on q. Let $S_{k,q}$ and $T_{k,q}$ denote respectively the left hand and right hand sides of equation (7).

If we write $S_{k,q}$ as a sum of two terms, the first in which $i_k < q$ and the second when $i_k = q$, we see that $S_{k,q}$ satisfies the recurrence relation

$$S_{k,q} = S_{k,q-1} + (x^{-q}+x^{q})^{-1}(x^{-q-1}+x^{q+1})^{-1}S_{k-1,q-2}$$

Thus we have only to show that $T_{k,q}$ satisfies the same relation. Now

$$T_{k,q-1} = \frac{\prod_{i=k}^{2k-1}(x^{-(q-i)} - x^{q-i})}{\prod_{i=1}^{k}(x^{-2i} - x^{2i})\prod_{i=0}^{k-1}(x^{-(q-i)} + x^{q-i})}$$

and

$$T_{k-1,q-2} = \frac{\prod_{i=k}^{2k-2}(x^{-(q-i)} - x^{q-i})}{\prod_{i=1}^{k-1}(x^{-2i} - x^{2i})\prod_{i=1}^{k-1}(x^{-(q-i)} + x^{q-i})}$$

The proof of the recurrence relation for $T_{k,q}$ is thus reduced to verifying that

$$\begin{aligned}&\frac{(x^{-(q-(k-1)} - x^{q-(k-1)})(x^{-(q-(k-1)} + x^{q-(k-1)})}{(x^{-q-1} + x^{q+1})(x^{-q} + x^{q})} \\ &= \frac{x^{-(q-(2k-1))} - x^{q-(2k-1)}}{x^{-q} + x^{q}} + \frac{x^{-2k} - x^{2k}}{(x^{-q} + x^{q})(x^{-q-1} + x^{q+1})}\end{aligned}$$

□

Theorem 3.5. *For $k \geq 3$ and $q \geq 2k-1$,*

$$\begin{aligned}&(x^{-1} + x^{1})^{-1}(x^{-2} + x^{2})^{-1}(x^{-q} + x^{q})^{-1}(x^{-q-1} + x^{q+1})^{-1} \\ &\quad\times \sum_{i_1=3}^{q-2(k-2)}\sum_{i_2=i_1+2}^{q-2(k-3)}\cdots\sum_{i_{k-2}=i_{k-3}+2}^{q-2}\prod_{j=1}^{k-2}(x^{-i_j} + x^{i_j})^{-1}(x^{-i_j-1} + x^{i_j+1})^{-1} \\ (8)\quad &= \frac{(x^{-k+1} - x^{k-1})(x^{-k} - x^{k})\prod_{i=k+1}^{2k-2}(x^{-q+i} - x^{q-i})}{\prod_{i=1}^{k}(x^{-2i} - x^{2i})\prod_{i=-1}^{k-2}(x^{-q+i} + x^{q-i})}\end{aligned}$$

Proof. Let us denote the left and right hand sides of the identity by $S_{k,q}$ and $T_{k,q}$ respectively. By Corollary 3.3 $S_{3,q} = T_{3,q}$. We write $S_{k,q}$ as the sum of two terms: in the first $i_{k-2} < q-2$ and in the second $i_{k-2} - q - 2$. As in the proof of the previous theorem we obtain a recurrence relation, in this case:

$$\begin{aligned}S_{k,q} &= S_{k,q-1}(x^{-q-1} + x^{q+1})^{-1}(x^{-q+1} + x^{q-1}) \\ &\quad + S_{k-1,q-2}(x^{-q} + x^{q})^{-1}(x^{-q-1} + x^{q+1})^{-1}\end{aligned}$$

It is routine to verify that $T_{k,q}$ satisfies the same recurrence relation. □

Subtracting equation (8) from equation (7) yields.

Corollary 3.6.

$$\begin{aligned}&\sum_{i_1=1}^{q-2(k-1)}\sum_{i_2=i_1+2}^{q-2(k-2)}\cdots\sum_{i_k=i_{k-1}+2}^{q}\prod_{j=1}^{k}(x^{-i_j} + x^{i_j})^{-1}(x^{-i_j-1} + x^{i_j+1})^{-1} \\ &\quad - (x^{-1} + x^{1})^{-1}(x^{-2} + x^{2})^{-1}(x^{-q} + x^{q})^{-1}(x^{-q-1} + x^{q+1})^{-1} \\ &\times \sum_{i_1=3}^{q-2(k-2)}\sum_{i_2=i_1+2}^{q-2(k-3)}\cdots\sum_{i_{k-2}=i_{k-3}+2}^{q-2}\prod_{j=1}^{k-2}(x^{-i_j} + x^{i_j})^{-1}(x^{-i_j-1} + x^{i_j+1})^{-1} \\ (9)\quad &= \frac{(x^{q} - x^{-q})\prod_{i=k+1}^{2k-2}(x^{-q+i} - x^{q-i})}{\prod_{k=1}^{q}(x^{2i} - x^{-2i})\prod_{i=-1}^{k-2}(x^{-q+i} + x^{q-i})}\end{aligned}$$

Proof of theorem 2.5 We recall that $1 \leq p < q$ and p and q are relatively prime. We set $\theta = p/q$ and $a_j = 2\cos(2\pi j\theta)$. We shall split the proof into two cases.

Case 1: $q \not\equiv 0 \pmod 4$. When $q \not\equiv 0 \pmod 4$ $a_j \neq 0$ for all j; moreover when $x = e^{2\pi i\theta}$, $x^{4i} \neq 1$ and $x^{2(q-i)} \neq -1$ for all i. Thus the denominator on the right hand side of (9) does not vanish but the numerator does. Hence by Theorem 3.1

$$\sum_{I \in \widetilde{S}\left[{q \atop k}\right]} (a_{I^c})^{-1} = 0$$

Upon multiplying by $a_1 a_2 \cdots a_q$ we obtain that

$$\sum_{I \in \widetilde{S}\left[{q \atop k}\right]} a_I = a_1 a_2 \cdots a_q \sum_{I \in \widetilde{S}\left[{q \atop k}\right]} (a_{I^c})^{-1} = 0.$$

Case 2: $q \equiv 0 \pmod 4$. Again we wish to show that $\sum_{I \in \widetilde{S}\left[{q \atop k}\right]} a_I = 0$ and so we must multiply both sides of equation (9) by $\prod_{i=1}^{q}(x^{-i} + x^i)$ and evaluate at $x = e^{2\pi i\theta}$.

The denominator of the right hand side of (9) is zero when $x^{4q} = 1$ or $x^{2(q-j)} = -1$, i.e. when $i = j = q/4$; the corresponding factors are $x^{-q/2} - x^{q/2}$ and $x^{-3q/4} + x^{3q/4}$ respectively.

Apart from the factor $x^{-q} - x^q$, the numerator of the right hand side of equation (9) is zero only when $x^{2(q-i)} = 1$, i.e. when $i = q/2$. This produces the factor $x^{-q/2} - x^{q/2}$ which cancels one of the zeros in the denominator. The other zero is cancelled when we multiply by $\prod_{i=1}^{q}(x^{-i}+x^i)$. Hence the product of $\prod_{i=1}^{q}(x^{-i}+x^i)$ and the right side of (9) is zero when $x = e^{2\pi i\theta}$. $\square$

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Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, T2T 1A1

E-mail address: mikel@math.ucalgary.ca

Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, K7L 3N6

E-mail address: mingo@mast.queensu.ca